

Existence and uniqueness of minimizers of general least gradient problems

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Abstract

Motivated by problems arising in conductivity imaging, we prove existence, uniqueness, and comparison theorems - under certain sharp conditions - for minimizers of the general least gradient problem

$$\inf_{u \in BV_f(\Omega)} \int_{\Omega} \varphi(x, Du),$$

where $f : \partial\Omega \rightarrow \mathbb{R}$ is continuous,

$$BV_f(\Omega) := \{v \in BV(\Omega) : \forall x \in \partial\Omega, \lim_{r \rightarrow 0} \operatorname{ess\,sup}_{y \in \Omega, |x-y| < r} |f(x) - v(y)| = 0\}$$

and $\varphi(x, \xi)$ is a function that, among other properties, is convex and homogeneous of degree 1 with respect to the ξ variable. In particular we prove that if $a \in C^{1,1}(\Omega)$ is bounded away from zero, then minimizers of the weighted least gradient problem $\inf_{u \in BV_f} \int_{\Omega} a |Du|$ are unique in $BV_f(\Omega)$. We construct counterexamples to show that the regularity assumption $a \in C^{1,1}$ is sharp, in the sense that it can not be replaced by $a \in C^{1,\alpha}(\Omega)$ with any $\alpha < 1$.

1 Introduction

Let Ω be a bounded open set in \mathbb{R}^n with Lipschitz boundary and $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function satisfying the following conditions.

(C_1) There exists $\alpha > 0$ such that $\alpha|\xi| \leq \varphi(x, \xi) \leq \alpha^{-1}|\xi|$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

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(C₂) $\xi \mapsto \varphi(x, \xi)$ is a norm for every x .

For any $u \in BV_{loc}(\mathbb{R}^n)$ let $\varphi(x, Du)$ denote the measure defined by

$$\int_A \varphi(x, Du) = \int_A \varphi(x, v^u(x)) |Du| \quad \text{for } A \text{ a bounded Borel set,} \quad (1)$$

where $|Du|$ is the total variation measure associated to the vector-valued measure Du , and v^u denotes the Radon-Nikodym derivative $v^u(x) = \frac{dDu}{d|Du|}$. (The right-hand side of (1) makes sense, since v^u is $|Du|$ -measurable, and hence $\varphi(x, v^u(x))$ is as well.) Standard measure-theory considerations and facts about BV functions imply that (see [1]) if U is an open set, then

$$\int_U \varphi(x, Du) = \sup \left\{ \int_U u \nabla \cdot Y dx \quad : \quad Y \in C_c^\infty(U; \mathbb{R}^n), \quad \sup \varphi^0(x, Y(x)) \leq 1 \right\}, \quad (2)$$

where $\varphi^0(x, \cdot)$ denotes the norm on \mathbb{R}^n dual to $\varphi(x, \cdot)$, defined by

$$\varphi^0(x, \xi) := \sup \{ \xi \cdot p : \varphi(x, p) \leq 1 \}.$$

For $u \in BV(\Omega)$, $\int_\Omega \varphi(x, Du)$ is called the φ -total variation of u in Ω . Also if A, E are subsets of \mathbb{R}^n , with A Borel and E having finite perimeter, then we shall write $P_\varphi(E; A)$ to denote the φ -perimeter of E in A , defined by

$$P_\varphi(E; A) := \int_A \varphi(x, D\chi_E), \quad (3)$$

where χ_E is the characteristic function of E . We will also write $P_\varphi(E)$ to mean $P_\varphi(E; \mathbb{R}^n)$. We remark that if ∂E is smooth enough, then

$$P_\varphi(E; A) := \int_{\partial E \cap A} \varphi(x, \nu_E(x)) d\mathcal{H}^{n-1} \quad \nu_E := \text{outer unit normal},$$

which is a generalized inhomogeneous, anisotropic area of ∂E in A .

In this paper we present existence, comparison, and uniqueness results for minimizers of the general least gradient problem

$$\inf_{v \in BV_f(\Omega)} \int_\Omega \varphi(x, Dv) \quad (4)$$

where $f \in C(\partial\Omega)$ and

$$BV_f(\Omega) := \{v \in BV(\Omega) : \forall x \in \partial\Omega, \lim_{r \rightarrow 0} \operatorname{ess\,sup}_{y \in \Omega, |x-y| < r} |f(x) - v(y)| = 0 \}.$$

We will prove existence of a minimizer in $BV_f(\Omega)$ of the general least gradient problem (4), as long as $\partial\Omega$ satisfies a positivity condition on a sort of generalized mean curvature related to the integrand φ . We refer to this as the *barrier condition*, and we defer its statement until later (see Definition 3, at the beginning of Section 3). We will prove the following existence result.

Theorem 1.1 *Suppose that $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function that satisfies $C_1 - C_2$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. If Ω satisfies the barrier condition with respect to φ , then for every $f \in C(\partial\Omega)$, the general least gradient problem (4) has a minimizer in $BV_f(\Omega)$.*

In fact we prove something slightly stronger; see Remark 3.4.

For our comparison and uniqueness results we do not need to assume the barrier condition. On the other hand, we require stronger convexity and regularity conditions on φ than we have so far assumed. In particular we will assume:

(C_3) $\varphi \in W_{loc}^{2,\infty}$ away from $\{\xi = 0\}$, and there exists $C > 0$ such that $\varphi_{\xi_i \xi_j}(x, \xi) p^i p^j \geq C|p'|^2$ for all $\xi \in S^{n-1}$ and $p \in \mathbb{R}^n$, where $p' := p - (p \cdot \xi)\xi$.

(C_4) φ and $D_\xi \varphi$ are $W^{2,\infty}$ away from $\xi = 0$, and there are positive constants ρ and λ such that

$$\begin{aligned} \varphi(x, \xi) + |D_\xi \varphi(x, \xi)| + |D_\xi^2 \varphi(x, \xi)| + |D_\xi^3 \varphi(x, \xi)| + \rho |D_x D_\xi \varphi(x, \xi)| \\ + \rho |D_x D_\xi^2 \varphi(x, \xi)| + \rho^2 |D_x^2 D_\xi \varphi(x, \xi)| \leq \lambda \end{aligned} \quad \text{for all } x \in \Omega, \xi \in S^{n-1}. \quad (5)$$

These conditions are needed for a result, due to Schoen and Simon [18], about partial regularity of φ -minimizing sets, which we state as Theorem 2.6. This result will play a crucial role in our uniqueness proof in Section 4. In addition, condition (C_3) is crucial also in Lemma 4.1.

The following theorem is our main uniqueness result. It also gives a stability estimate (with best constant) for the solutions with respect to errors in the boundary data.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, and assume $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $C_1 - C_4$. Suppose that u_1, u_2 minimize (4) in $BV_{f_1}(\Omega)$ and $BV_{f_2}(\Omega)$ respectively, for $f_1, f_2 \in C(\partial\Omega)$. Then*

$$|u_2 - u_1| \leq \sup_{\partial\Omega} |f_2 - f_1| \quad \text{a.e. in } \Omega. \quad (6)$$

Moreover

$$u_2 \geq u_1 \quad \text{a.e. in } \Omega, \text{ if } f_2 \geq f_1 \text{ on } \partial\Omega. \quad (7)$$

In particular, for every $f \in C(\partial\Omega)$, there is at most one minimizer of (4) in $BV_f(\Omega)$.

The following regularity result, valid only in low dimensions, is obtained by essentially the same arguments as in the proof of Theorem 1.2.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, and assume $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $C_1 - C_4$. If $n = 2$ or 3 and u minimizes (4) in $BV_f(\Omega)$ for $f \in C(\partial\Omega)$, then u is continuous in $\bar{\Omega}$.*

For the special case $\varphi(x, \xi) = a(x)|\xi|$, the problem (4) is the weighted least gradient problem

$$\inf_{u \in BV_f(\Omega)} \int_{\Omega} a |Du|. \quad (8)$$

In [20], assuming a barrier condition on $\partial\Omega$ (see (3.1) and (3.2) in [20]), the authors showed that when $a \equiv 1$, problem (8) admits a unique minimizer, and this minimizer is continuous in every dimension $n \geq 2$. Their proof is valid with very little change for problem (8) when a is smooth enough and bounded below. It is not valid, however, for functionals φ not arising from a Riemannian metric, and it is not clear, and would be hard to determine, exactly how much regularity their proof requires of a . The different approach we follow here yields uniqueness results and continuity in low dimensions, under *sharp* regularity hypotheses, for a larger class of problems, without any barrier condition on $\partial\Omega$. We do not know whether, in the setting of Theorem 1.2 with $n \geq 4$, it is possible for a minimizer to be discontinuous.

We now briefly describe the conductivity imaging problem that leads to (8). Let $\sigma(x)$ be a positive function that models inhomogeneous isotropic conductivity of a body Ω . If u is the electric potential corresponding to the voltage f on the boundary of Ω , then u solves the Dirichlet problem

$$\nabla \cdot (\sigma \nabla u) = 0, \quad u|_{\partial\Omega} = f.$$

By Ohm's law, the corresponding current density is $J = -\sigma \nabla u$. Consider the inverse problem of determining σ from knowledge of $|J|$ inside Ω (with a known f prescribed on $\partial\Omega$). Such internal data can be obtained using Magnetic Resonance Imaging [2]. It was first shown in [16] that the corresponding voltage potential u is the unique solution of the weighted least gradient problem

$$\operatorname{argmin} \left\{ \int_{\Omega} a |Dv| : u \in W^{1,1}(\Omega) \cap C(\bar{\Omega}) \text{ and } u|_{\partial\Omega} = f \right\}, \quad (9)$$

with $a = |J|$ given. This uniqueness result has recently been extended to $u \in BV(\Omega)$ [12]. Once u is determined (see [13] for a convergent numerical algorithm) the computation of σ is straightforward. The uniqueness results in [16] and [12] assume that the weight a is of the form $a = |J|$, as described above, but are valid for weights $a \in C^\alpha(\Omega)$ and allow a to vanish in certain sets. The following direct consequence of Theorem 1.2 provides uniqueness, comparison, and stability results for general weights a which are not necessarily of the form $a = |J|$. We do however need more restrictive assumptions on a .

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary. Suppose $a \in C^{1,1}(\Omega)$ is positive and bounded away from zero, and $u_1, u_2 \in BV(\Omega)$ minimize (8) in $BV_{f_1}(\Omega), BV_{f_2}(\Omega)$ respectively, for some $f_1, f_2 \in C(\partial\Omega)$. Then*

$$|u_2 - u_1| \leq \sup_{\partial\Omega} |f_2 - f_1| \quad \text{a.e. in } \Omega. \quad (10)$$

Moreover

$$u_2 \geq u_1 \text{ a.e. in } \Omega, \text{ if } f_2 \geq f_1 \text{ on } \partial\Omega. \quad (11)$$

In particular, (8) has at most one minimizer in $BV_f(\Omega)$, and any minimizer is continuous if $n \leq 3$.

In fact, in the setting of Theorem 1.4 (as well as Theorem 1.5 below) minimizers are continuous for $\Omega \subset \mathbb{R}^n$ as long as $n \leq 7$, see Remark 4.5.

Our next result shows that the regularity assumption $a \in C^{1,1}(\Omega)$, and hence also the regularity assumptions C_3, C_4 in Theorem 1.2, are in a sense sharp.

Proposition 1.1 *For any $\alpha < 1$, there exists a bounded smooth domain $\Omega \subset \mathbb{R}^n$ with connected boundary, $f \in C(\partial\Omega)$, and a function $a \in C^{1,\alpha}(\Omega)$ with $\inf_{x \in \Omega} a(x) > 0$ such the weighted least gradient problem (8) has infinitely many minimizers in $BV_f(\Omega)$.*

In fact, since the weighted gradient functional is convex, if uniqueness fails, then there must be infinitely many minimizers.

The minimizers constructed in Proposition 1.1 are all discontinuous, so in this sense the Proposition also shows that our regularity assumptions on φ are sharp in Theorem 1.3.

It is also easy to see that all of our uniqueness and comparison results can fail if $\partial\Omega$ is not connected, even if $a \in C^\infty(\bar{\Omega})$. For example, if Ω is an annulus $B(2,0) \setminus \overline{B(1,0)} \subset \mathbb{R}^n$ and $a(x) = |x|^{1-n}$, then any function of the form $u(x) = g(|x|)$ minimizes (8) with respect to its boundary data, as long as g is monotone. Here and throughout the paper $B(r,x)$ denotes the open ball of radius r centred at x .

Recently in [7] authors presented a method for recovering the conformal factor of an anisotropic conductivity matrix in a known conformal class from one interior measurement. Assume that the matrix valued conductivity $\sigma(x)$ is of the form

$$\sigma(x) = c(x)\sigma_0(x)$$

where $c(x) \in C^\alpha(\Omega)$ is a positive scalar valued function and $\sigma_0 \in C^\alpha(\Omega, \text{Mat}(n, \mathbb{R}^n))$ is a known positive definite symmetric matrix valued function. In medical imaging σ_0 can be determined using Diffusion Tensor Magnetic Resonance Imaging (see [9] and the references therein). In [12] the authors showed that the corresponding voltage potential u is the unique solution of the least gradient problem

$$\operatorname{argmin}\left\{\int_{\Omega} \varphi(x, Dv) : u|_{BV(\Omega)}, \quad u|_{\partial\Omega} = f\right\},$$

where φ is given by

$$\varphi(x, \xi) = a(x) \left(\sum_{i,j=1}^n \sigma_0^{ij}(x) \xi_i \xi_j \right)^{1/2}, \quad (12)$$

$$a = \sqrt{\sigma_0^{-1} J \cdot J}, \quad (13)$$

and J is the current density vector field generated by imposing the voltage f at $\partial\Omega$. Once u is determined the function $c(x)$ can easily be calculated. This uniqueness result assumes that the weight a is of the form (13) and it applies for weights $a \in C^\alpha(\Omega)$ that may vanish in certain sets (see [7]). The following immediate corollary of Theorem 1.2 provides uniqueness, comparison, and stability results for general weights of the form (12) with a not necessarily of the form (13), but requires more restrictive assumptions on a .

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, and assume $\varphi(x, \xi)$ is given by (12), where $a \in C^{1,1}(\Omega)$ is positive and bounded away from zero and $\sigma_0 \in C^{1,1}(\Omega, \text{Mat}(n, \mathbb{R}))$ satisfies*

$$m|\xi|^2 \leq \sum_{i,j=1}^n \sigma_0^{ij} \xi_i \xi_j \leq M|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n,$$

for some $0 < m, M < \infty$. If $u_1, u_2 \in BV(\Omega)$ minimize (8) in $BV_{f_1}(\Omega), BV_{f_2}(\Omega)$ respectively, with $f_1, f_2 \in C(\partial\Omega)$, then

$$|u_2 - u_1| \leq \sup_{\partial\Omega} |f_2 - f_1| \quad \text{a.e. in } \Omega. \quad (14)$$

Moreover

$$u_2 \geq u_1 \quad \text{a.e. in } \Omega, \text{ if } f_2 \geq f_1 \text{ on } \partial\Omega. \quad (15)$$

In particular, for the class of φ as described above, (4) has at most one minimizer in $BV_f(\Omega)$, and any minimizer is continuous if $n \leq 3$.

The paper is organized as follows. In Section 2 we will present some preliminary results and basic facts about φ -total variation. Sections 3 and 4 are devoted to the proof of the existence and uniqueness results, respectively. In Section 5, we shall prove Proposition 1.1 by constructing a one parameter family of minimizers of (8). Finally, in Section 6, we provide a more convenient formulation of the barrier condition when the boundary of Ω is sufficiently smooth.

2 Preliminary results

In this section we develop some basic facts about φ -total variation, defined in (1) above. These facts are well-known for the usual notion of (isotropic, homogeneous) total variation, and we sketch some proofs here only to make it clear that the standard arguments are still valid in the somewhat more general setting that we consider here. The paper [1] is a good reference for φ -total variation.

It follows easily from (2) that

$$\int_A \varphi(x, D(u_1 + u_2)) \leq \int_A \varphi(x, Du_1) + \int_A \varphi(x, Du_2) \quad (16)$$

if A is open, and hence if A is any Borel set.

The definition (1) and the Fleming-Rishel coarea formula for BV functions imply the following coarea formula for the φ -total variation.

Proposition 2.1 (Remark 4.4 in [1]) *If $u \in BV_{loc}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$ is a bounded Borel set, then*

$$\int_A \varphi(x, Du) = \int_{-\infty}^{+\infty} P_\varphi(X_t; A) dt, \quad (17)$$

where $X_t := \{x \in \Omega : u(x) > t\}$ and P_φ is as defined in equation (3).

It is a straightforward consequence of the coarea formula that for any $u \in BV(\Omega)$ and $\lambda \in \mathbb{R}$, if we write $u_1 := \max(u - \lambda, 0)$ and $u_2 = u - u_1$, then

$$\int_{\Omega} \varphi(x, Du) = \int_{\Omega} \varphi(x, Du_1) + \int_{\Omega} \varphi(x, Du_2). \quad (18)$$

Lemma 2.1 *Let $A \subset \mathbb{R}^n$ be a Borel set and $E_1, E_2 \subset \mathbb{R}^n$ be of locally finite perimeter with respect φ . Then*

$$P_{\varphi}(E_1 \cup E_2; A) + P_{\varphi}(E_1 \cap E_2; A) \leq P_{\varphi}(E_1; A) + P_{\varphi}(E_2; A). \quad (19)$$

Proof: We apply (18) with $u = \chi_{E_1} + \chi_{E_2}$ and $\lambda = 1$. Then $u_1 = \chi_{E_1 \cap E_2}$, and $u_2 = \chi_{E_1 \cup E_2}$. It follows that

$$\begin{aligned} \int_A \varphi(x, D\chi_{E_1 \cup E_2}) ds + \int_A \varphi(x, D\chi_{E_1 \cap E_2}) ds &\stackrel{(18)}{=} \int_A \varphi(x, D(\chi_{E_1} + \chi_{E_2})) \\ &\stackrel{(16)}{\leq} \int_A \varphi(x, D\chi_{E_1}) + \int_A \varphi(x, D\chi_{E_2}). \end{aligned}$$

Rewriting in terms of P_{φ} yields (19). □

Definition 1 (i) *We say that a function $u \in BV(\mathbb{R}^n)$ is φ -total variation minimizing in a set $\Omega \subset \mathbb{R}^n$ if*

$$\int_{\mathbb{R}^n} \varphi(x, Du) \leq \int_{\mathbb{R}^n} \varphi(x, Dv) \quad \text{for all } v \in BV(\mathbb{R}^n) \text{ such that } u = v \text{ a.e. in } \Omega^c.$$

(ii) *Similarly, we say that $E \subset \mathbb{R}^n$ of finite perimeter is φ -area minimizing in Ω if*

$$P_{\varphi}(E) \leq P_{\varphi}(F) \quad \text{for all } F \subset \mathbb{R}^n \text{ such that } F \cap \Omega^c = E \cap \Omega^c \text{ a.e..}$$

We emphasize that in the definitions above, $u - v$ is not required to have compact support in Ω , and $E \setminus F$ is not required to be compactly contained in Ω .

If $v \in BV(\mathbb{R}^n)$ and Ω is an open set with Lipschitz boundary, we will write v^+ and v^- to denote the outer and inner trace of v on $\partial\Omega$. Recall that these are functions in $L^1(\partial\Omega; \mathcal{H}^{n-1})$, characterized by the fact that for \mathcal{H}^{n-1} almost every $x \in \partial\Omega$,

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho^n} \int_{B_{\rho}(x) \setminus \Omega} |v^+(x) - v(y)| dy = \lim_{\rho \rightarrow 0} \frac{1}{\rho^n} \int_{B_{\rho}(x) \cap \Omega} |v^-(x) - v(y)| dy = 0. \quad (20)$$

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, with Lipschitz boundary. Given $g \in L^1(\partial\Omega; \mathcal{H}^{n-1})$, define*

$$I_{\varphi}(v; \Omega, g) := \int_{\partial\Omega} \varphi(x, \nu_{\Omega}) |g - v^-| d\mathcal{H}^{n-1} + \int_{\Omega} \varphi(x, Dv).$$

where ν_{Ω} denotes the outer unit normal to Ω . Then $u \in BV(\mathbb{R}^n)$ is φ -total variation minimizing in Ω if and only if $u|_{\Omega}$ minimizes $I_{\varphi}(\cdot; \Omega, g)$ for some g , and moreover $g = u^+$.

Proof: We recall some basic properties of traces. First, if $v \in BV(\mathbb{R}^n)$ then v^+ and v^- are \mathcal{H}^{n-1} integrable on $\partial\Omega$, and conversely, for every $g \in L^1(\partial\Omega; \mathcal{H}^{n-1})$ there exists some $v \in BV(\mathbb{R}^n)$ such that $g = v^+$ say. Second, we note that

$$\int_{\partial\Omega} \varphi(x, Dv) = \int_{\partial\Omega} \varphi(x, v^v) |Dv| = \int_{\partial\Omega} \varphi(x, \nu_\Omega) |v^+ - v^-| d\mathcal{H}^{n-1}. \quad (21)$$

To see this, note that $|Dv|$ can only concentrate on a set of dimension $n - 1$ if that set is a subset of the jump set of v , so (21) follows from standard descriptions of the jump part of Dv .

Now if $u, v \in BV(\mathbb{R}^n)$ satisfy $u = v$ a.e. in Ω^c , then it follows from (1) that $\int_{\bar{\Omega}^c} \varphi(x, Du) = \int_{\bar{\Omega}^c} \varphi(x, Dv)$. In addition, $u^+ = v^+$, so using (21) we deduce that

$$\int_{\mathbb{R}^n} \varphi(x, Du) - \int_{\mathbb{R}^n} \varphi(x, Dv) = I_\varphi(u; \Omega, u^+) - I_\varphi(v; \Omega, u^+).$$

The lemma easily follows. \square

Lemma 2.3 *Let Ω be a bounded Lipschitz domain and let $E_1, E_2 \subset \mathbb{R}^n$ be area minimizing in Ω . If $E_1 \cap \Omega^c \subset E_2 \cap \Omega^c$, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are area minimizing in Ω .*

Proof: For $i = 1, 2$, let $\mathcal{A}_i := \{F \subset \mathbb{R}^n : F \cap \Omega^c = E_i \cap \Omega^c\}$. Our hypotheses imply that $E_1 \cap E_2 \in \mathcal{A}_1$ and $E_1 \cup E_2 \in \mathcal{A}_2$, so it suffices to show that

$$P_\varphi(E_1 \cap E_2) \leq \inf_{F \in \mathcal{A}_1} P_\varphi(F) = P_\varphi(E_1), \quad P_\varphi(E_1 \cup E_2) \leq \inf_{F \in \mathcal{A}_2} P_\varphi(F) = P_\varphi(E_2).$$

Since the opposite inequalities hold, these follow directly from (19). \square

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $u \in BV(\mathbb{R}^n)$ be φ -total variation minimizing in Ω . Let*

$$E_\lambda = \{x \in \mathbb{R}^n : u(x) \geq \lambda\}.$$

Then E_λ is φ -area minimizing in Ω for every λ .

A similar result used in earlier work about the case $\varphi(x, \xi) = |\xi|$ (see for example [20]) uses a somewhat different notion of minimizing, in which only perturbations with compact support in Ω are allowed. For the proof we will use the following lemma.

Lemma 2.5 *Assume u_k is φ -total variation minimizing in Ω for $k \geq 1$ and*

$$u_k \rightarrow u \text{ in } L^1(\Omega), \text{ and } u_k^\pm \rightarrow u^\pm \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}).$$

Then u is φ -total variation minimizing in Ω .

Proof: It follows from (2) via quite standard arguments that

$$\int_{\Omega} \varphi(x, Du) \leq \liminf_k \int_{\Omega} \varphi(x, Du_k), \quad (22)$$

and this, with the L^1 convergence of the traces, implies that

$$I_{\varphi}(u; \Omega, u^+) \leq \liminf_{k \rightarrow \infty} I_{\varphi}(u_k; \Omega, u_k^+). \quad (23)$$

Now for any $v \in BV(\mathbb{R}^n)$ such that $u = v$ a.e. in Ω^c ,

$$\begin{aligned} I_{\varphi}(u_k; \Omega, u_k^+) &\leq I_{\varphi}(v; \Omega, u_k^+) \\ &\leq I_{\varphi}(v; \Omega, u^+) + \int_{\partial\Omega} \varphi(x, \nu_{\Omega}) |u^+ - u_k^+| d\mathcal{H}^{n-1} \\ &\leq I_{\varphi}(v; \Omega, u^+) + \alpha^{-1} \int_{\partial\Omega} |u^+ - u_k^+| d\mathcal{H}^{n-1} \end{aligned}$$

using Lemma 2.2, the minimality of u_k and standing assumption C_1 . It follows from this, (23), and again the L^1 convergence of the traces that $I_{\varphi}(u; \Omega, u^+) \leq I_{\varphi}(v; \Omega, u^+)$, which proves the proposition. \square

Proof of Theorem 2.4: Our argument is modelled on the proof of Theorem 1 in [3].

For $\lambda \in \mathbb{R}$, let $u_1 = \max(u - \lambda, 0)$, $u_2 = u - u_1$. Let $g \in BV(\mathbb{R}^n)$ with $\text{supp}(g) \subset \bar{\Omega}$. Since u is a minimizer,

$$\begin{aligned} \int_{\Omega} \varphi(x, Du_1) + \int_{\Omega} \varphi(x, Du_2) &\stackrel{(18)}{=} \int_{\Omega} \varphi(x, Du) \\ &\leq \int_{\Omega} \varphi(x, D(u + g)) \\ &\stackrel{(16)}{\leq} \int_{\Omega} \varphi(x, D(u_1 + g)) + \int_{\Omega} \varphi(x, Du_2). \end{aligned}$$

Hence u_1 is a minimizer. Repeating the same argument, one verifies that

$$\chi_{\epsilon, \lambda} := \min\left(1, \frac{1}{\epsilon} u_1\right) = \begin{cases} 0 & \text{if } u \leq \lambda \\ \epsilon^{-1}(u - \lambda) & \text{if } \lambda \leq u \leq \lambda + \epsilon \\ 1 & \text{if } u \geq \lambda + \epsilon \end{cases}$$

is also a minimizer of (1). It is clear that for a.e. $\lambda \in \mathbb{R}$,

$$\mathcal{L}^n(\{x \in \Omega : u(x) = \lambda\}) = \mathcal{H}^{n-1}(\{x \in \partial\Omega : u^{\pm}(x) = \lambda\}) = 0, \quad (24)$$

and it is straightforward to check, using (20), that if (24) holds, then

$$\chi_{\epsilon, \lambda} \rightarrow \chi_{\lambda} := \chi_{E_{\lambda}} \text{ in } L^1_{loc}(\mathbb{R}^n), \quad \chi_{\epsilon, \lambda}^{\pm} \rightarrow \chi_{\lambda}^{\pm} \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}). \quad (25)$$

Thus Lemma 2.5 implies that $\chi_{E_{\lambda}}$ is φ -total variation minimizing in Ω , and hence that E_{λ} is φ -area minimizing in Ω .

If λ does not satisfy (24), then let λ_k be an increasing sequence such that $\lambda_k \rightarrow \lambda$ and λ_k satisfies (24) for every k . Then one can check that

$$\chi_{\lambda_k} \rightarrow \chi_\lambda \quad \text{in } L^1_{loc}(\mathbb{R}^n), \quad \chi_{\lambda_k}^\pm \rightarrow \chi_\lambda^\pm \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}).$$

as $k \rightarrow \infty$, so it again follows from Proposition 2.5 that E_λ is φ -area minimizing in Ω . \square

Definition 2 *Let $E \subset \mathbb{R}^n$. A point $x \in \partial E$ is called a regular point if there exists $\rho > 0$ such that $\partial E \cap B(x, \rho)$ is a C^2 hypersurface. We denote the set of all regular points of ∂E by $\text{reg}(\partial E)$. We will say that x is a singular point of ∂E if $x \in \text{sing}(\partial E)$, where*

$$\text{sing}(\partial E) = \partial E \setminus \text{reg}(\partial E).$$

If E is a measurable subset of \mathbb{R}^n , we will write

$$E^{(1)} := \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(B(r, x) \cap E)}{\mathcal{H}^n(B(r))} = 1\}. \quad (26)$$

The following singularity estimate is due to Schoen, Simon, and Almgren [18, Theorem I.3.1 and Corollary I.3.2], and plays a crucial role in our uniqueness proof in Section 4.

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^n$. Suppose $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $C_1 - C_4$. If E is φ -area minimizing in Ω , then*

$$\begin{cases} \mathcal{H}^{n-3}(\text{sing}(\partial E^{(1)}) \cap \Omega) < \infty & \text{if } n \geq 4 \\ \text{sing}(\partial E^{(1)}) \cap \Omega = \emptyset, & \text{if } n \leq 3. \end{cases} \quad (27)$$

One reason for considering $\partial E^{(1)}$ rather than ∂E is that the former is insensitive to modifications of E on sets of measure 0.

Remark 2.7 *In [18], the set that we have identified as $\partial E^{(1)}$, for a φ -area minimizing set E , is described in a slightly different way, but standard facts about BV functions imply that our description is equivalent in the context of the theorem.*

Remark 2.8 *Some ambiguous wording in [18] suggests that Theorem 2.6 might require that $\varphi \in W^{3,\infty}_{loc}$ away from $\{\xi = 0\}$, in addition to the hypotheses in C_4 . However, inspection of the proof shows that C_4 is all the regularity that is needed for φ . Indeed, regularity of φ is used in the proof of [18, Theorem I.3.1] in the following ways:*

1. to ensure $C^{2,\gamma}_{loc}$ regularity for weak solutions $w : B_r^{n-1} \rightarrow \mathbb{R}$ of equations of the form

$$-\sum_{i=1}^{n-1} \partial_{x_i}(\varphi_{\xi_i}(x, w, -Dw, 1)) - \varphi_{x_n}(x, w, -Dw, 1) = 0, \quad x \in B_r^{n-1}.$$

2. as a hypothesis for basic ϵ -regularity results for φ -area minimizing currents.

3. for maximum principle arguments such as those in Lemma 4.1 below
4. in estimates such as those in the proof of [18, Lemma I.2.5].

For the first point listed above, it is rather standard that the regularity assumed in C_4 suffices, together with the structural conditions $C_1 - C_3$, and for the second, it is proved in the reference [17] cited in [18] that these hypotheses are enough. For the maximum principle argument, it is clear from our proof of Lemma 4.1 below (which is just a slightly more detailed version of an argument from [18]) that $C_1 - C_3$ suffice. As well, all the estimates in [18] involve only the derivatives appearing in C_4 , and yield constants that depend only on the dimension n and the constants ρ, λ in C_4 .

3 Existence

In this section we prove Theorem 1.1. First we give a precise statement of the geometric condition that is a main hypothesis of the theorem.

Definition 3 Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function that satisfies $C_1 - C_2$. We say that Ω satisfies the barrier condition if for every $x_0 \in \partial\Omega$ and $\epsilon > 0$ sufficiently small, if V minimizes $P_\varphi(\cdot; \mathbb{R}^n)$ in

$$\{W \subset \Omega : W \setminus B(\epsilon, x_0) = \Omega \setminus B(\epsilon, x_0)\}, \quad (28)$$

then

$$\partial V^{(1)} \cap \partial\Omega \cap B(\epsilon, x_0) = \emptyset.$$

In the case $\varphi(x, \xi) = |\xi|$, the barrier condition is equivalent, at least for smooth sets, to the one introduced in [20].

A convenient interpretation of the barrier condition, if $\partial\Omega$ is sufficiently smooth, is provided by the following result. See also Remark 3.2 below.

Lemma 3.1 Assume that $\partial\Omega$ is C^2 and that φ satisfies $C_1 - C_3$. Define the signed distance $d(\cdot)$ to $\partial\Omega$ by

$$d(x) := \begin{cases} \text{dist}(x, \partial\Omega) & \text{if } x \in \Omega \\ -\text{dist}(x, \partial\Omega) & \text{if not.} \end{cases}$$

Then Ω satisfies the barrier condition if

$$-\sum_{i=1}^n \partial_{x_i} \varphi_{\xi_i}(x, Dd(x)) > 0 \quad \text{on a dense subset of } \partial\Omega. \quad (29)$$

The proof of Lemma 3.1 is given in Section 6. Although we do not prove it, if φ and Ω satisfy the above hypotheses, then the barrier condition is in fact *equivalent* to (29).

Remark 3.2 If $\partial\Omega$ is defined (locally) as a graph $x_n = w(x')$ of a C^2 function w (so that locally $\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > w(x')\}$) then an equivalent formulation of (29) can be seen from the following equality

$$\sum_{i=1}^n \partial_{x_i} \varphi_{\xi_i}(x, Dd) = \sum_{j=1}^{n-1} \partial_{x_j} \varphi_{\xi_j}(x', w, -Dw, 1) + \varphi_{x_n}(x', w, -Dw, 1), \quad (30)$$

at points $x = (x', w(x'))$ on $\partial\Omega$. For the convenience of the readers we include a proof of (30) in Section 6.

Our main use of the barrier condition is the following technical lemma.

Lemma 3.3 Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain that satisfies the barrier condition with respect to φ , and assume that $E \subset \mathbb{R}^n$ is φ -area minimizing in Ω . Then

$$\{x \in \partial\Omega \cap \partial E^{(1)} : B(\epsilon, x) \cap \partial E^{(1)} \subset \bar{\Omega} \text{ for some } \epsilon > 0\} = \emptyset.$$

Proof: Assume there exists $x_0 \in \partial\Omega \cap \partial E^{(1)}$ such that $B(\epsilon, x_0) \cap \partial E^{(1)} \subset \bar{\Omega}$ for some $\epsilon > 0$ and let V be a minimizer of $P_\varphi(\cdot; \mathbb{R}^n)$ in (28). Existence of such a set V is standard, for reasons discussed in the proof of Theorem 1.1 below.

Then it follows from Lemma 2.3 that $V' = V \cup (E \cap \Omega)$ also minimizes $P_\varphi(\cdot; \mathbb{R}^n)$ in (28). It is easy to see that $x_0 \in \partial V'^{(1)}$. Hence

$$x_0 \in \partial V'^{(1)} \cap \partial\Omega \cap B(\epsilon, x_0) \neq \emptyset.$$

This contradicts the barrier condition and finishes the proof. \square

Since $f \in C(\partial\Omega)$, it can be extended to a function in $C(\Omega^c)$ and throughout the paper we will denote this extension to $C(\Omega^c)$ by f again. Now we are ready to prove our existence result.

Proof of Theorem 1.1: Without loss of generality we may assume that $f \in BV(\mathbb{R}^n)$, since basic trace theorems guarantee that every \mathcal{H}^{n-1} integrable function on Ω is the trace of some (continuous) function in $BV(\Omega^c)$. Define

$$\mathcal{A}_f := \{v \in BV(\mathbb{R}^n) : v = f \text{ on } \Omega^c\},$$

and note that $BV_f(\Omega) \hookrightarrow \mathcal{A}_f$, in the sense that any element v of $BV_f(\Omega)$ is the restriction to Ω of a unique element of \mathcal{A}_f . Thus it suffices to prove that the functional

$$F(v) := \int_{\mathbb{R}^n} \varphi(x, Dv) dx.$$

has a minimizer $u \in \mathcal{A}_f$, and that u can be identified with an element of $BV_f(\Omega)$.

Existence of a minimizer u is standard, as F is coercive in $BV(\mathbb{R}^n)$ (a consequence of C_1) and weakly lower semicontinuous, as already noted in (22), and because $BV(\mathbb{R}^n) \hookrightarrow L^1_{loc}$.

We next use the barrier condition to show that $u \in BV_f(\Omega)$. If not, there exists some $x \in \partial\Omega$ and $\delta > 0$ such that

$$\operatorname{ess\,sup}_{y \in \Omega, |x-y| < r} (f(x) - u(y)) \geq \delta \quad \text{or} \quad \operatorname{ess\,sup}_{y \in \Omega, |x-y| < r} (u(y) - f(x)) \geq \delta \quad (31)$$

for every $r > 0$. Assume that the latter condition holds. It follows from this, along with the definition of \mathcal{A}_f and the continuity of f , that $x \in \partial E^{(1)}$ for $E := E_{f(x)+\delta/2}$. Recall that E is φ -area minimizing in Ω by Theorem 2.4. However, since f is continuous in Ω^c and $u \in \mathcal{A}_f$, it is clear that $u < f(x) + \delta/2$ in $B(\varepsilon, x) \setminus \Omega$ for all sufficiently small ε . But Lemma 3.3 shows that this is impossible.

If the first alternative in (31) holds, then we define $E := \{y \in \mathbb{R}^n : u(y) \leq f(x) - \delta/2\}$ and find in the same way that $u \in BV_f(\Omega)$.

Finally, note that if $v \in BV_f$, then the inner and outer traces of v both equal f at every point of $\partial\Omega$, and so it follows from (21) that that $|Dv|(\partial\Omega) = 0$. Hence, if $v \in BV_f(\Omega)$, then

$$\begin{aligned} F(v) &= \int_{\Omega} \varphi(x, Dv) + \int_{\partial\Omega} \varphi(x, Dv) + \int_{\mathbb{R}^n \setminus \Omega} \varphi(x, Dv) \\ &= \int_{\Omega} \varphi(x, Dv) + \int_{\mathbb{R}^n \setminus \Omega} \varphi(x, Df) \end{aligned}$$

So the fact that $u \in BV_f(\Omega)$ minimizes F in \mathcal{A}_f implies that it minimizes (4) in $BV_f(\Omega)$. \square

Remark 3.4 *The above arguments show that if $u \in BV_f(\Omega) \hookrightarrow \mathcal{A}_f$ minimizes (4) in $BV_f(\Omega)$, then it also is φ -total variation minimizing in Ω , so that $F(u) \leq F(v)$ for all $v \in \mathcal{A}_f \supsetneq BV_f(\Omega)$. This conclusion does not require the barrier condition.*

4 Uniqueness and continuity

In this section we prove the comparison principle and uniqueness result stated in Theorem 1.2. We shall use the following results from dimension theory. The proofs can be found in [8] (Chapter IV).

Proposition 4.1 *Let U be an open set in a connected k -dimensional manifold which is neither empty nor dense. Then the topological dimension of ∂U is $k - 1$.*

Proposition 4.2 *A connected k -dimensional manifold can not be disconnected by a subset of dimension $k - 2$.*

For a proof of the following proposition see [8] (Chapter VIII, §4).

Proposition 4.3 *Let X be a metric space. Then the Hausdorff dimension of X is bounded below by its topological dimension.*

Let $E \subset \mathbb{R}^n$ be a φ -area minimizing set and y_0 be a regular point of $\partial E^{(1)}$. Then for ρ sufficiently small, we can arrange, after a suitable choice of coordinates, that

$$\partial E^{(1)} \cap B(\rho, y_0) = \{(y, w(y)) : y \in A\},$$

for some open $A \subset \mathbb{R}^{n-1}$ and $w \in C^2(A)$. Moreover, by rewriting $\int \varphi(x, D\chi_E)$ in terms of w and computing the first variation, we find that w satisfies

$$\varphi_{x_n}(y, w, -Dw, 1) + \sum_{i=1}^{n-1} \frac{d}{dx_i}(\varphi_{\xi_i}(y, w, -Dw, 1)) = 0, \quad y \in A. \quad (32)$$

The following strong maximum principle is a standard consequence of basic elliptic regularity results.

Lemma 4.1 *Suppose φ satisfies $C_1 - C_3$. Assume also that w_1 and w_2 are C^2 solutions of (32) on a $(n-1)$ -dimensional ball $B(y_0, \rho)$ such that $w_1 \leq w_2$, and that $w_1 = w_2$ at some point in $B(y_0, \rho)$. Then $w_1 = w_2$ on $B(y_0, \rho)$.*

Note that C_4 is not needed here. The failure of Lemma 4.1 for less smooth integrands φ is the mechanism behind the counterexamples presented in the next section. In order to make it clear where the regularity assumptions on φ are used we therefore present some details of the proof, which however is quite standard (see *e.g.* [18, Lemma 2.4]).

Proof: Let $w = w_2 - w_1$. If we write $\mathcal{D}(w)$ for the left-hand side of (32), then by rewriting the identity $0 = \mathcal{D}(w_2) - \mathcal{D}(w_1) = \int_0^1 \frac{d}{ds} \mathcal{D}(sw_2 + (1-s)w_1) ds$, we find that w satisfies the equation

$$\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i} (-\langle \varphi_{\xi_i \xi_j} \rangle w_{x_j} + \langle \varphi_{\xi_i x_n} \rangle w) + \langle \varphi_{x_n \xi_i} \rangle w_{x_i} - \langle \varphi_{x_n x_n} \rangle w = 0, \quad (33)$$

where we use the notation

$$\langle g \rangle(y) := \int_0^1 g(y, w^s(y), -Dw^s(y), 1) ds, \quad w^s = sw_2 + (1-s)w_1.$$

It follows from our assumptions that all the coefficients in the above equation are L^∞ , and that $\langle \varphi_{\xi_i \xi_j} \rangle$ is positive definite. Since $w \geq 0$ and w vanishes at some point in $B(y_0, \rho)$, it follows from Moser's Harnack inequality, which is valid for equation (33), that $w \equiv 0$ in $B(y_0, \rho)$. \square

We shall also need the following lemma.

Lemma 4.2 *Let Ω be a bounded Lipschitz domain with connected boundary and assume that $E \subset \mathbb{R}^n$ is φ -area minimizing in Ω . If R is a nonempty connected component of $\text{reg}(\partial E^{(1)}) \cap \Omega$, then $\bar{R} \cap \partial \Omega \neq \emptyset$.*

Recall that $\text{reg}(\partial E^{(1)})$ denotes the regular part of $\partial E^{(1)}$ (see Definition 2). The idea of the proof is that if the conclusion fails, then we could modify E in a way that decreases its φ -perimeter without changing $E \cap \Omega^c$, either by deleting a component of E or by “filling in a hole”; this would contradict the minimality of E . We defer the full proof to the end of the section.

We will deduce Theorems 1.2 and 1.3 from the following geometric comparison principle, which is of interest in its own right.

Theorem 4.3 (Comparison principle for φ -area minimizing sets) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, and suppose that φ satisfies $C_1 - C_4$. Assume that $E_1, E_2 \subset \mathbb{R}^n$ are φ -area minimizing sets in Ω . and also that*

$$E_1 \setminus \Omega \subset\subset E_2 \setminus \Omega. \quad (34)$$

If Ω satisfies the barrier condition, or if

$$\partial E_1^{(1)} \setminus E_2^{(1)} \subset \Omega, \quad \text{and} \quad \partial E_2^{(1)} \cap \overline{E_1^{(1)}} \subset \Omega, \quad (35)$$

then

$$E_1^{(1)} \subset E_2^{(1)}.$$

Moreover, if $n \leq 3$ then $E_1^{(1)} \subset\subset E_2^{(1)}$.

Remark 4.4 *When applying the above theorem in the proof of our main uniqueness result, Theorem 1.2, assumption (35) will be satisfied as a consequence of the sense in which the minimizers u_1, u_2 assume their boundary values f_1, f_2 . That is why the barrier condition is not needed for our uniqueness results.*

Proof of Theorem 4.3: We may assume that E_i is open, $i = 1, 2$, since otherwise we may replace E_i by $\text{int } E_i^{(1)}$, which in view of Theorem 2.6 differs from $E_i^{(1)}$, and hence E_i , on a set of measure zero. Also, if $E_i = \text{int } E_i^{(1)}$, then clearly $\partial E_i = \partial E_i^{(1)}$, and $E^{(1)} \subset E^{(2)}$ if $E_1 \subset E_2$. So in the sequel we may drop all superscripts on $E_i^{(1)}$, $i = 1, 2$ (but not for example on $(E_1 \cup E_2)^{(1)}$, since in general $\partial(E_1^{(1)} \cup E_2^{(1)}) \neq \partial(E_1 \cup E_2)^{(1)}$.)

It also suffices to prove the theorem under hypothesis (35), since this follows from the barrier condition if $E_1 \setminus \Omega \subset\subset E_2 \setminus \Omega$. Indeed, when (34) holds it is clear that $\partial E_1 \setminus E_2 \subset \overline{\Omega}$. And if $x_0 \in (\partial E_1 \setminus E_2) \cap \partial \Omega$, then there exists $\epsilon_0 > 0$ such that $\partial E_1 \cap B(\epsilon, x_0) \subset \overline{\Omega}$ for all $\epsilon < \epsilon_0$, because otherwise $x_0 \in (\overline{E_1 \setminus \Omega}) \setminus E_2$ which violates the assumption $E_1 \setminus \Omega \subset\subset E_2 \setminus \Omega$. On the other hand, if the barrier condition holds, then according to Lemma 3.3, it cannot be the case that $\partial E_1 \cap B(\epsilon, x_0) \subset \overline{\Omega}$. The proof of the other inclusion in (35) is essentially identical.

We now assume toward a contradiction that $E_1 \not\subset E_2$ (Figure 1). Note that Theorem 2.6 implies that $E_i \cap \Omega$ differs from its interior by a set of measure zero, so it follows that $E_1 \setminus E_2$ has nonempty interior. We claim that then

$$\dim_{\text{Haus}}(\partial E_1 \cap \partial E_2) \geq n - 2. \quad (36)$$

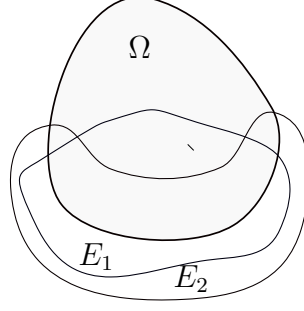


Figure 1: Failure of the comparison principle

We will assume that $\mathcal{H}^{n-1}(\partial E_1 \cap \partial E_2) = 0$, since otherwise (36) is immediate. Then either $\partial E_1 \setminus \overline{E_2}$ or $\partial E_2 \cap E_1$ must have positive \mathcal{H}^{n-1} measure, since $\partial(E_1 \setminus E_2)$ is an $(n-1)$ -dimensional set (by Proposition 4.1), and

$$\partial(E_1 \setminus E_2) \subset (\partial E_1 \setminus \overline{E_2}) \cup (E_1 \cap \partial E_2) \cup (\partial E_1 \cap \partial E_2).$$

For concreteness we assume that $\mathcal{H}^{n-1}(\partial E_1 \setminus \overline{E_2}) > 0$; the other case is essentially the same.

Since $\partial E_1 \setminus \overline{E_2} \subset \Omega$ by (35), and since Theorem 2.6 guarantees that \mathcal{H}^{n-3} a.e. point of $\partial E_1 \cap \Omega$ is regular, we may fix a connected component R of $\text{reg}(\partial E_1) \cap \Omega$ such that $R \setminus \overline{E_2}$ is nonempty.

It follows from Lemma 4.2 that $\bar{R} \cap \partial \Omega \neq \emptyset$, and then $\bar{R} \cap \partial \Omega \subset \partial E_1 \cap \partial \Omega \subset E_2$ by (35). Thus $\bar{R} \cap E_2 \neq \emptyset$. Since E_2 is open, it follows that $R \cap E_2 \neq \emptyset$.

Then $R \cap \partial E_2$ separates R into two nonempty components, $R \setminus \overline{E_2}$ and $R \cap E_2$. Since the definition of a regular point implies that R is an $(n-1)$ -dimensional submanifold of \mathbb{R}^n , and R is connected by definition, it follows from Proposition 4.2 that the topological dimension of $R \cap \partial E_2$ is at least $n-2$. Since $R \cap \partial E_2 \subset \partial E_1 \cap \partial E_2$, claim (36) now follows from Proposition 4.3.

It follows from (36) and Theorem 2.6 that

$$\mathcal{H}^{n-2}(\text{reg}(\partial E_1) \cap \text{reg}(\partial E_2)) > 0. \quad (37)$$

Note also that

$$\begin{aligned} \text{reg}(\partial E_1) \cap \text{reg}(\partial E_2) \subset & [\partial(E_1 \cup E_2)^{(1)} \cap \partial(E_1 \cap E_2)^{(1)}] \\ & \cup [\partial(E_1 \cup (\overline{E_2})^c)^{(1)} \cap \partial(E_1 \cap (\overline{E_2})^c)^{(1)}]. \end{aligned} \quad (38)$$

Indeed, if $x_0 \in \text{reg}(\partial E_1) \cap \text{reg}(\partial E_2)$, then by the definition of a regular point, we may write both boundaries ∂E_1 and ∂E_2 locally as graphs over the same domain in the same hyperplane, say of C^2 functions w_1 and w_2 . We may also assume that E_1 lies below the graph of w_1 near x_0 . It is straightforward to verify that if E_2 also lies below the graph of w_2 near x_0 , then

$$x_0 \in \partial(E_1 \cup E_2)^{(1)} \cap \partial(E_1 \cap E_2)^{(1)}$$

whereas if E_2 lies above the graph of w_2 near x_0 , then

$$x_0 \in \partial(E_1 \cup (\overline{E_2})^c)^{(1)} \cap \partial(E_1 \cap (\overline{E_2})^c)^{(1)}.$$

By Corollary 2.3, all of the sets $E_1 \cup E_2$, $E_1 \cap E_2$, \dots on the right-hand side of (38) are φ -area minimizing on Ω , and hence their boundaries are all regular in Ω away from a set of dimension at most $n - 3$. It follows that for a suitable choice $F_1 = E_2$ or \bar{E}_2^c , (36) and (27) that

$$U := \{x \in \partial E_1 \cap \partial E_2 : \partial(E_1 \cup F_1)^{(1)} \text{ and } \partial(E_1 \cap F_1)^{(1)} \text{ are regular at } x\} \quad (39)$$

satisfies $\mathcal{H}^{n-2}(U) > 0$. Let R be a connected component of $\text{reg}(\partial E_1)$ that intersects U , and let

$$R_0 := R \setminus S, \quad S := \text{sing}(\partial(E_1 \cup F_1)^{(1)}) \cup \text{sing}(\partial(E_1 \cap F_1)^{(1)}).$$

We claim that $R_0 \subset U$. To prove this, first note that the topological dimension of S is bounded by $\dim_{\text{Haus}}(S) \leq n - 3$, so Proposition 4.2 implies that R_0 is connected. Thus it suffices to show that $U \cap R_0$ is nonempty, open and closed in R_0 . It follows from the definitions $\emptyset \neq U \cap R \subset U \cap R_0$, and also that $U \cap R_0 = R_0 \cap \partial E_2$, which is clearly closed. So we only need to check openness. For this, fix $x \in R_0 \cap U$, and note that since $(E_1 \cap F_1) \subset (E_1 \cup F_1)$ and both boundaries are regular at x , we may write both boundaries locally as C^2 graphs over the same domain in the same hyperplane, say of functions w_1 and w_2 such that $w_1 \leq w_2$ in their domain. Then Lemma 4.1 implies that $\partial(E_1 \cap F_1)$ coincides with $\partial(E_1 \cup F_1)$ in a neighbourhood of x , which implies that E_1 coincides with F_1 in a neighborhood of x and hence that ∂E_1 and $\partial E_2 = \partial F_1$ coincide in a neighborhood of x . Thus $U \cap R_0$ is open.

Now the dimension estimate of S implies that R_0 is dense in R , and thus $\bar{R} = \bar{R}_0 \subset \bar{U}$. We deduce using Lemma 4.2, that $\partial E_1 \cap \partial E_2 \cap \partial \Omega$ is nonempty, which is impossible due to (35). Thus we have arrived at a contradiction.

Finally, suppose that $n \leq 3$. We already know that $E_1 \subset E_2$, and if E_1 is not compactly contained in E_2 , then $\partial E_1 \cap \partial E_2 \cap \Omega$ is nonempty. Since $n \leq 3$, ∂E_1 and ∂E_2 are both regular everywhere in Ω , so we can invoke Lemma 4.1 to find that $U := \partial E_1 \cap \partial E_2$ is open in both ∂E_1 and ∂E_2 . Then arguing as above, we find that $\partial U \cap \Omega$ is nonempty, and this again is a contradiction. \square

Now we use Theorem 4.3 to establish our main uniqueness and continuity results.

Proof of Theorem 1.2: We first prove (7). As before we extend f_i for $i = 1, 2$ to continuous functions on $\bar{\Omega}^c$ (still denoted by f_i). We may assume that $f_1 \leq f_2$ in Ω^c , since otherwise we may replace f_1, f_2 by $\min(f_1, f_2)$ and $\max(f_1, f_2)$ respectively. For $i = 1, 2$, we extend u_i to a function (still denoted u_i) on \mathbb{R}^n by setting it equal to f_i on Ω^c .

Suppose toward a contradiction that (7) is not true. Then since

$$\{x \in \Omega : u_1(x) > u_2(x)\} = \bigcup_{(\lambda_1, \lambda_2) \in \mathbb{Q} \times \mathbb{Q}} \{x \in \Omega : u_1(x) \geq \lambda_1 > \lambda_2 > u_2(x)\}$$

there must be some rational numbers $\lambda_1 > \lambda_2$ such that

$$\mathcal{H}^n(\{x \in \Omega : u_1(x) \geq \lambda_1 > \lambda_2 > u_2(x)\}) > 0.$$

If we define

$$E_i := \{x \in \mathbb{R}^n : u_i(x) \geq \lambda_i\}.$$

then this says exactly that $\mathcal{H}^n(E_1 \setminus E_2) > 0$.

On the other hand, it follows from the definition of $BV_{f_i}(\Omega)$ and the continuity of f that if $x \in \partial E_i^{(1)} \cap \partial\Omega$, then $f_i(x) = \lambda_i$. Since $f_1 \leq f_2$ on $\partial\Omega$ and $\lambda_1 > \lambda_2$, the definitions imply that (35) holds. Hence it follows from Theorem 4.3 that $E_1^{(1)} \subset E_2^{(1)}$, and therefore that $\mathcal{H}^n(E_1 \setminus E_2) = 0$. Since this contradicts the above, we conclude that (7) holds.

Finally, as is well known, it is easy to deduce (6) from (7). For example, to prove that $u_1 - u_2 \leq \sup_{\partial\Omega} |f_2 - f_1|$ a.e. in Ω , we apply (7) to u_1 and $\tilde{u}_2 := u_2 + \sup_{\partial\Omega} |f_2 - f_1|$. The opposite inequality is proved by the same argument. \square

Remark 4.5 For $\varphi(x, \xi) = a(x)|\xi|$, or more generally φ of the form (8), minimizers are continuous in $\Omega \subset \mathbb{R}^n$ for $n \leq 7$. The point is that in this case the boundary of a φ -area minimizing set is actually a minimal hypersurfaces with respect to some Riemannian metric, and as such has better regularity properties than in the case of the more general class of integrands we consider in this paper. In particular, if E is φ -area minimizing in $\Omega \subset \mathbb{R}^n$ and $n \leq 7$, then $\text{sing}(\partial E^{(1)}) \cap \Omega = \emptyset$. (This is documented for example in [14].) Continuity for $n \leq 7$ is established by using this fact in place of Theorem 2.6 and repeating the above arguments.

Proof of Theorem 1.3: As in Theorem 1.2, we set u equal to f on Ω^c . For $x \in \Omega$ we define

$$u^*(x) := \lim_{r \rightarrow 0} \text{ess sup}_{B(r,x)} u, \quad u_*(x) := \lim_{r \rightarrow 0} \text{ess inf}_{B(r,x)} u,$$

We must show that $u^* = u_*$ everywhere in Ω . Assume toward a contradiction that this fails, so that

$$u_*(x_0) < \lambda_2 < \lambda_1 < u^*(x_0)$$

for some $x_0 \in \Omega$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Define

$$E_i := \{x \in \mathbb{R}^n : u(x) \geq \lambda_i\}$$

As in the proof of Theorem 1.2, the hypotheses of Theorem 4.3 are satisfied, and (since now $n \leq 3$) it follows that $E_1^{(1)} \subset\subset E_2^{(1)}$, and hence that $\partial E_1^{(1)} \cap \partial E_2^{(1)} = \emptyset$.

So to arrive at a contradiction, it suffices to check that $x_0 \in \partial E_i^{(1)}$ for $i = 1, 2$. This is straightforward. In fact, since $u^*(x_0) > \lambda_i$, every ball around x_0 contains a subset of E_i of positive measure, and hence a subset of $E_i^{(1)}$ of (the same) positive measure. Similarly, since $u_*(x_0) < \lambda_i$, every ball around x_0 contains a subset of $(E_i^1)^c$ of positive measure, and it follows that $x_0 \in \partial E_i^{(1)}$ for $i = 1, 2$. \square

Finally, we conclude this section with the proof of Lemma 4.2, which played an important role in the above arguments.

Proof of Lemma 4.2: Let R be a nonempty connected component of $\text{reg}(\partial E^{(1)}) \cap \Omega$, for a set E that is φ area-minimizing in Ω . Also, assume toward a contradiction that $\bar{R} \cap \partial\Omega = \emptyset$.

The definition of a regular point implies that R is a C^2 submanifold of \mathbb{R}^n . Given a smooth $(n-1)$ -form ψ in \mathbb{R}^n , we will write

$$[[R]](\psi) := \int_R \psi$$

where R has the same orientation as ∂E , which in turn inherits its orientation from E in the standard way. And given an $(n-2)$ -form η , we will write

$$\partial[[R]](\eta) := [[R]](d\eta). \quad (40)$$

(In particular, here “ ∂ ” denotes the boundary in a distributional sense, rather than the topological boundary.) We first claim that

$$\partial[[R]] = 0. \quad (41)$$

Toward this end, note that Stokes’ Theorem implies that $\partial[[R]](\eta) = 0$ if the support of η does not intersect $\overline{R} \setminus R$. Since $\overline{R} \subset \Omega$, the definitions imply that $\overline{R} \setminus R \subset \partial E \setminus \text{reg}(\partial E) = \text{sing}(\partial E) =: S$. Thus $\partial[[R]]$ is supported in S . Since $\mathcal{H}^{n-2}(S) = 0$, the claim (41) follows from standard geometric measure theory considerations, which we summarize as follows:

- It is clear that $[[R]]$ is a $(n-1)$ -dimensional rectifiable current and hence a $(n-1)$ -dimensional flat chain; see [5, 4.1.24].
- Thus $\partial[[R]]$ is a $(n-2)$ -dimensional flat chain, see [5, 4.1.12].
- As an $(n-2)$ -dimensional flat chain whose support has \mathcal{H}^{n-2} measure zero, $\partial[[R]]$ must be trivial, see [5, 4.1.20 and 2.10.6]. This is (41).

Next we claim that there exists a set $F \subset \subset \Omega$ of finite perimeter such that

$$[[R]] = \pm \partial[[F]] \quad (42)$$

where, on the right-hand side, $[[F]]$ denotes the n -current corresponding to integration over F , and ∂ is defined as in (40). (The meaning of $[[\cdot]]$ should always be clear from the context.) Indeed, the fact that R is bounded and $\partial[[R]] = 0$ implies that there exists some compactly supported integer-valued BV function u_R such that

$$[[R]](\psi) = \int_{\mathbb{R}^n} u_R d\psi \quad \text{for every smooth compactly supported } (n-1)\text{-form } \psi. \quad (43)$$

This follows for example as an easy special case of the isoperimetric theorem, see [5, 4.2.10]. One can deduce from (43) and the coarea formula (or for a detailed proof see [19, Theorem 27.6]) that

$$[[R]] = \sum_{k=1}^{\infty} \partial[[\{x : u_R(x) \geq k\}]] - \sum_{k=1}^{\infty} \partial[[\{x : u_R(x) \geq -k\}]] \quad (44)$$

and

$$\mathcal{H}^{n-1}(R) = \sum_{k=1}^{\infty} \mathcal{H}^{n-1}(\partial\{x : u_R(x) \geq k\}) + \sum_{k=1}^{\infty} \mathcal{H}^{n-1}(\partial\{x : u_R(x) \leq -k\}). \quad (45)$$

However, it is shown in [5, 4.2.25] that the fact that R is a C^2 , connected submanifold of \mathbb{R}^n implies that there can only be one nontrivial term on the right-hand side of (44). (The point is that a smooth connected submanifold of \mathbb{R}^n is *indecomposable*, which means exactly that it admits no nontrivial decomposition (44) satisfying (45).) So u_R is the characteristic function of a set F , up to a sign, and (43) thus reduces to the claim (42). We must also show that $F \subset\subset \Omega$. To see this, we observe from (43) that $Du_R = 0$ in the sense of distributions away from R , and in particular in Ω^c . The fact that $\partial\Omega$ is connected implies that Ω^c is connected. Hence, since it has compact support, $u_R = 0$ in Ω^c , so that $F \subset \bar{\Omega}$. Moreover, since $\bar{R} \cap \partial\Omega = \emptyset$ and $\bar{R}, \partial\Omega$ are both compact, these sets are separated by a positive distance. Hence, reasoning as above we see that $u_R = 0$ in a neighborhood of $\partial\Omega$, which means that $F \subset\subset \Omega$.

By duality between $(n-1)$ -forms and vector fields, (42) is equivalent to

$$\int_R \eta \cdot \nu_E d\mathcal{H}^{n-1} = \pm \int_F \nabla \cdot \eta d\mathcal{L}^n \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n),$$

and in view of the smoothness of R , this implies that

$$\bar{R} = \partial F^{(1)}, \quad \nu_E = \pm \nu_F \text{ on } R. \quad (46)$$

First assume for concreteness that $\nu_E = \nu_F$ on R , and define $\tilde{E} := E \setminus F$. Then $E \cap \Omega^c = \tilde{E} \cap \Omega^c$. We will show that $P_\varphi(\tilde{E}) < P_\varphi(E)$, contradicting the minimality of E . We will use the notation

$$E^{(0)} := \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(B(r, x) \cap E)}{\mathcal{H}^n(B(r))} = 0\}, \quad \partial_* E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$$

and the fact that for any set E of finite perimeter, $\partial_* E$ is \mathcal{H}^{n-1} measurable, an approximate unit normal ν_E exists \mathcal{H}^{n-1} a.e. in $\partial_* E$, and

$$P_\varphi(E) := \int_{\partial_* E} \varphi(x, \nu_E) d\mathcal{H}^{n-1}.$$

For the sets E, F above, the regularity of R implies that $\partial_* E = \partial E^{(1)}$ in Ω , up to sets of dimension $n-3$, and similarly for F . In addition,

$$\partial_* \tilde{E} = \partial_*(E \setminus F) = (\partial_* E \cap F^{(0)}) \cup (E^{(1)} \cap \partial_* F) \cup \{x \in \partial_* E \cap \partial_* F : \nu_E = -\nu_F\}$$

up to sets of \mathcal{H}^{n-1} measure zero. The content of this statement can be understood by drawing a picture, and a proof can be found for example in [10], Theorem 16.3. Then since $\nu_E = \nu_F$ on R , it follows that

$$\partial_* \tilde{E} = \partial_* E \cap F^{(0)} \subset \partial_* E \setminus R \subset \partial_* E, \quad (47)$$

up to \mathcal{H}^{n-1} null sets. Then we complete the proof by calculating

$$P_\varphi(E) = \int_{\partial_* E} \varphi(x, \nu_E) d\mathcal{H}^{n-1} > \int_{\partial_* E \setminus R} \varphi(x, \nu_E) d\mathcal{H}^{n-1} \geq \int_{\partial_* \tilde{E}} \varphi(x, \nu_{\tilde{E}}) d\mathcal{H}^{n-1} = P_\varphi(\tilde{E}),$$

If $\nu_E = -\nu_F$ on R then we define $\tilde{E} := E \cup F$ and argue in essentially the same way. □

5 Non-uniqueness

In this section we show that the regularity assumptions in our uniqueness theorems are sharp. Indeed for any $\alpha < 1$ and $n \geq 2$, we will construct a $C^{1,\alpha}$ function a on a bounded region $\Omega \subset \mathbb{R}^n$ such that the weighted least gradient problem (4) has infinitely many minimizers in $BV(\Omega)$ (Proposition 1.1).

The main ingredient in the proof is the following Lemma, in which we simultaneously construct both a family of functions u_σ and a vector field that calibrates them. In the lemma, we will write points in \mathbb{R}^n in the form $(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Lemma 5.1 *Let $D := \{(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < 3\}$. For $0 \leq \sigma \leq 1$, there exist a function $u_\sigma \in BV(D)$ and a vector field $J \in C^{\frac{1+\alpha}{2}}(D; \mathbb{R}^n)$ such that*

$$\nabla \cdot J = 0 \quad \text{in } \mathcal{D}'(D), \quad |J| \in C^{1,\alpha}(D), \quad (48)$$

$$\int_D J \cdot Du_\sigma = \int_D |J| |Du_\sigma|, \quad (49)$$

and $|J| > 0$ in D .

We will need not just that $\nabla \cdot J$ vanishes in the sense of distributions, but actually a slightly stronger property. This will be established later, in the proof of the Proposition, see (57), so we do not carefully check this condition here.

Proof: For $(x, z) \in D$, we will write $r := |x|$. We will take J to have the form

$$J(x, z) = \left(\frac{-x}{|x|^{n-1}} \psi_z(x, z), \frac{1}{|x|^{n-2}} \psi_r(|x|, z) \right), \quad (50)$$

for a function $\psi(r, z)$ to be chosen, satisfying

$$\psi(r, z) = r^{n-1} \quad \text{if } r \leq \frac{1}{8} \quad (51)$$

as well as other conditions to be stated later. Such vector fields always satisfy $\nabla \cdot J = 0$ in the sense of distributions, provided ψ is regular enough, which will be the case here. (See (57) below.) So the main point is now simply (49). To arrange that this holds, we will also take u to depend only on r and z , in which case (49) reduces to the condition that level curves of u_σ in the r - z plane are orthogonal to $(-\psi_z, \psi_r)$, or equivalently, are parallel to $\nabla \psi$. This is an ordinary differential equation for level curves of u_σ , and if $\nabla \psi$ is not Lipschitz continuous, solutions need not be unique. Thus failure of uniqueness on the level of ODEs will be the basis for failure of uniqueness for the variational problem.

Our first task is thus to choose ψ so that uniqueness fails for certain flows along the vector field $\nabla \psi$. We find it easier to write down a (mostly) explicit example, chosen to facilitate our later construction of u_σ , than to proceed by abstract arguments. So in addition to (51), we require that

$$\psi(r, z) = r - g(r) \left(\frac{1}{1+\theta} \right) |z|^{1+\theta} \quad \text{if } r \geq \frac{1}{4} \quad (52)$$

for $\theta < 1$ to be fixed below, where g is a smooth function such that

$$g(r) = \begin{cases} 0 & \text{if } r \leq \frac{1}{3}, \\ 1 & \text{if } r \in [\frac{1}{2}, 1], \\ -1 & \text{if } r \in [2, 3], \end{cases} \quad \text{and} \quad g(r) \geq 0 \quad \text{if } r \in (\frac{1}{3}, \frac{1}{2}).$$

We finally require that $\psi(r, z)$ is a smooth function of r alone when $r \leq \frac{1}{4}$, such that (51) holds and $\psi_r > 0$ for $0 < r \leq \frac{1}{4}$.

It is clear that J is smooth away from $\{z = 0, r \geq \frac{1}{4}\}$, and also that J is $C^{0,\theta}$. Also, for $r \geq \frac{1}{3}$,

$$|J| = r^{2-2n} |\nabla \psi| = r^{2-2n} (\psi_r^2 + g^2(r) z^{2\theta})^{1/2}$$

and since ψ_r^2 is C^2 and positive, it is straightforward to check that $|J| \in C^{1,2\theta-1}(D)$. To satisfy the conclusions of the lemma we choose $\theta = \frac{1+\alpha}{2}$.

Now we define a family of functions $u_\sigma(r, z)$ such that every level curve of every u_σ is an integral curve of $\nabla \psi$.

Toward this end, note that $\nabla \psi = (\psi_r, 0)$ when $z = 0$, with $\psi_r > 0$, so the r -axis is an integral curve of $\nabla \psi$. Notice also that $\nabla \psi = (1, \pm z^\theta)$ if $r \in [\frac{1}{2}, 1] \cup [2, 3]$, and so in these regions one can explicitly integrate to find integral curves of $\nabla \psi$. The following definition (in which ζ_1, ζ_2 will be defined below) is thus quite natural:

$$\begin{aligned} u_\sigma(r, z) &:= 0 && \text{if } z < 0, \\ u_\sigma(r, z) &= 1 && \text{if } \begin{cases} r \leq 1 \text{ and } z > \zeta_1(r) \\ 1 \leq r \leq 2 \text{ and } z > 0, \\ r \geq 2 \text{ and } z > \zeta_2(r) \end{cases} \\ u_\sigma(r, z) &:= \sigma && \text{if } r \leq 1 \text{ and } 0 < z < \zeta_1(r), \\ u_\sigma(r, z) &:= \frac{1}{1-\theta} z^{1-\theta} - r + 3 && \text{if } r \geq 2 \text{ and } 0 < z < \zeta_2(r). \end{aligned}$$

Figure 2 is a sketch of these regions in the (right half of the) $r - z$ plane.

To complete the definition, we need to specify $\zeta_1(r), \zeta_2(r)$. First, we choose ζ_2 such that u_σ is continuous on the set $\{(r, \zeta_2(r)) : 2 \leq r \leq 3\}$. Thus $\zeta_2(r)$ is characterized by the identity

$$\{(r, \zeta_2(r)) : 2 \leq r \leq 3\} := \{(r, z) : 2 \leq r \leq 3, \frac{1}{1-\theta} z^{1-\theta} - r + 3 = 1\}. \quad (53)$$

Second, we choose $\zeta_1(r)$ to be any function such that $r \in [0, 1] \mapsto (r, \zeta_1(r))$ is an integral curve of $\nabla \psi$ such that $\zeta_1(1) = 0$ and $\zeta_1 > 0$ somewhere in $(0, 1)$. For example, we choose ζ_1 to equal $[(1-\theta)(r-1)]^{1/(1-\theta)}$ for $\frac{1}{2} \leq r \leq 1$. Then $\zeta_2(r)$ for $0 \leq r \leq \frac{1}{2}$ may be found by solving the appropriate ODE, which is just $(1, \zeta_1'(r)) \cdot \nabla^\perp \psi(r, \zeta_1(r)) = 0$.

It remains to verify that $u_\sigma \in BV(D)$ and that (49) holds for every $\sigma \in [0, 1]$. Let us write ∇u_σ to denote the absolutely continuous part of Du_σ and Σ to denote the jump set of u_σ , which is independent of σ for $\sigma \in (0, 1)$. Recalling that $r = |x|$ for $x \in \mathbb{R}^{n-1}$, we have

$$\nabla u_\sigma = \begin{cases} (-\frac{x}{r}, z^{-\theta}) & \text{if } r \geq 2 \text{ and } 0 < z < \zeta_2(r) \\ 0 & \text{otherwise} \end{cases}$$

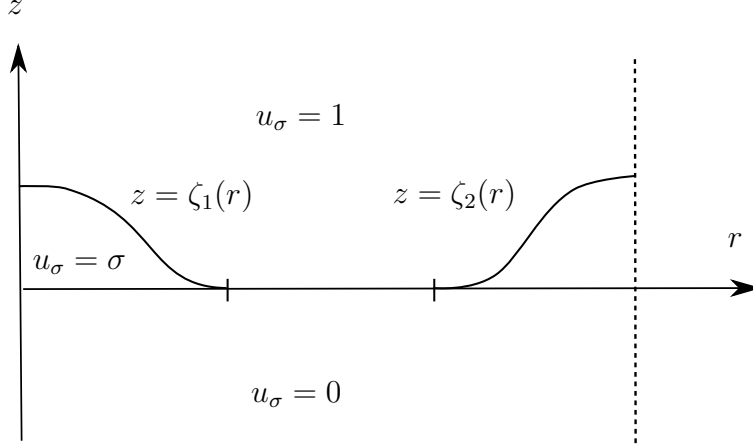


Figure 2: $u_\sigma(r, z)$

and $\Sigma \subset \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 = \{(x, z) \in D : z = 0\}, \quad \Sigma_2 := \{(x, z) \in D : r \leq 1, z = \zeta_1(r)\}. \quad (54)$$

(Indeed, $\Sigma = \Sigma_1$ if $\sigma = 1$ and Σ_2 if $\sigma = 0$, and otherwise $\Sigma = \Sigma_1 \cup \Sigma_2$.) It is easy to check that $\nabla u_\sigma \in L^1$ and since $u_\sigma \in L^\infty$ and the jump set has finite \mathcal{H}^{n-1} measure, we infer that $u_\sigma \in BV(D)$ for every σ , and

$$\int_D J \cdot Du_\sigma = \int_D J \cdot \nabla u_\sigma + \int_\Sigma J \cdot \nu [u_\sigma^+ - u_\sigma^-] d\mathcal{H}^{n-1}, \quad (55)$$

where ν is the upward unit normal to Σ and $u_\sigma^\pm(x, y) = \lim_{\epsilon \rightarrow 0^\pm} u_\sigma(x, y + \epsilon)$. We claim that

$$J \cdot \nabla u_\sigma = |J| |\nabla u_\sigma| \quad \text{in } D, \quad J \cdot \nu = |J| \quad \text{on } \Sigma. \quad (56)$$

Indeed, the first claim is a routine verification, as is the second for Σ_1 . For Σ_2 it follows in a straightforward way from the ODE solved by $\zeta_1(r)$.

Finally, since $u_\sigma^+ - u_\sigma^- \geq 0$ on Σ for every $\sigma \in [0, 1]$, we deduce from (55) and (56) that (49) holds. \square

We now show that the functions $(u_\sigma)_{\sigma \in [0, 1]}$ constructed above yield a counterexample to uniqueness for a suitable weight a .

Proof of Proposition 1.1: Let Ω be an open subset of D that contains the jump set $\Sigma = \Sigma_1 \cup \Sigma_2$, see (54). Then u_σ is continuous at $\partial\Omega$ and $u_\sigma|_{\partial\Omega} = u_0|_{\partial\Omega}$ for all $\sigma \in [0, 1]$. In particular, if we write f to denote the boundary value of u_0 , then $u_\sigma \in BV_f(\Omega)$ for every $\sigma \in [0, 1]$.

Next, let $a = |J|$ where J is the vector field constructed in Lemma 5.1. We now claim that that for every $w \in BV(\Omega)$, we have

$$\int_\Omega J \cdot Dw = \int_{\partial\Omega} w J \cdot \nu. \quad (57)$$

To see this, write $J = (J_1, J_2)$ with J_1 denoting the first $n-1$ components of J . Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that $\rho \equiv 0$ in $[-1, 1]$ and $\rho = 1$ for $|t| \geq 2$. Define $\rho_\epsilon(t) = \rho(\frac{t}{\epsilon})$ and

$$J_\epsilon(x, z) = (J_1(x, z)\rho_\epsilon(z), J_2(x, z)).$$

Then from the form (50) of J one checks that $J_\epsilon \in C^1(\Omega)$ and $\nabla \cdot J_\epsilon = 0$, so that

$$\int_{\Omega} J_\epsilon \cdot Dw = \int_{\partial\Omega} w J_\epsilon \cdot \nu. \quad \text{for every } w \in BV(\Omega).$$

Since ρ_ϵ is independent of x , letting $\epsilon \rightarrow 0$ we obtain (57). It follows that

$$\int_{\Omega} a |Du_\sigma| \stackrel{(49)}{=} \int_{\Omega} J \cdot Du_\sigma = \int_{\partial\Omega} J \cdot \nu u_0.$$

Now consider any $w \in BV_f(\Omega)$. Then $w|_{\partial\Omega} = f = u_0|_{\partial\Omega}$, so by (57) we obtain

$$\int_{\Omega} a |Dw| \geq \int_{\Omega} J \cdot Dw = \int_{\partial\Omega} J \cdot \nu u_0 = \int_{\Omega} a |Du_\sigma|$$

for all $\sigma \in [0, 1]$. Therefore for every $\sigma \in [0, 1]$, the function u_σ is a minimizer of weighted the least gradient problem (8). \square

6 About the barrier condition

This section is devoted the proof of the lemma, stated earlier, that provides a reformulation of the barrier condition for smooth domains.

Proof of Lemma 3.1: Assume toward a contradiction that (29) holds, but that the barrier condition fails. Then there exists $x_0 \in \partial\Omega$, a sequence $\varepsilon_k \rightarrow 0$, and sets $V_k \subset \Omega$ such that

$$V_k \text{ minimizes } P_\varphi(\cdot, \mathbb{R}^n) \text{ in } \{W \subset \Omega : W \setminus B(\varepsilon_k, x_0) = \Omega \setminus B(\varepsilon_k, x_0)\},$$

but

$$\partial V_k^{(1)} \cap \partial\Omega \cap B(\varepsilon_k, x_0) \neq \emptyset. \quad (58)$$

We will replace V_k by $V_k^{(1)}$ and drop the superscripts. By a change of coordinates, we may assume that $x_0 = 0$, and that $\nu(x_0) = (0, \dots, 0, 1)$, where ν denotes the outer unit normal to $\partial\Omega$. For each k we define

$$Z_k := \frac{1}{\varepsilon_k} V_k, \quad \Omega_k := \frac{1}{\varepsilon_k} \Omega, \quad \varphi_k(x, \xi) := \varphi(\varepsilon_k x, \xi).$$

Then by rescaling we find that

$$Z_k \text{ minimizes } P_{\varphi_k}(\cdot, \mathbb{R}^n) \text{ in } \{W \subset \Omega : W \setminus B(1) = \Omega_k \setminus B(1)\}.$$

It follows that, if $W \setminus B(1) = \Omega_k \setminus B(1)$, then $P_{\varphi_k}(Z_k, U) \leq P_{\varphi_k}(W; U)$ for any open set U such that $B(1) \subset\subset U$.

It is convenient to write points in \mathbb{R}^n in the form (x', x_n) with $x' \in \mathbb{R}^{n-1}$. We will also write $B'(r)$ to denote the open ball of radius r about the origin in \mathbb{R}^{n-1} . Then the smoothness of $\partial\Omega$ and our choice of coordinates imply that there exists a smooth function $\omega : B'(\delta) \rightarrow \mathbb{R}$ for some $\delta > 0$, such that near $x_0 = 0$, we can write $\partial\Omega$ as the graph of ω , and that $\nabla\omega(0) = 0$. If we define $\omega_k(x') = \frac{1}{\varepsilon_k}\omega(\varepsilon_k x')$, then for k sufficiently large,

$$\Omega_k \cap (B'(2) \times (-2, 2)) = \{(x', x_n) : x' \in B'(2), -2 < x_n < \omega_k(x')\}.$$

Note that $\omega_k \rightarrow 0$ in $C^2(B'(2))$ as $k \rightarrow \infty$. We will also write

$$A'_k := \{x' \in \mathbb{R}^{n-1} : (x', \omega_k(x')) \in B(1)\}, \quad A_k := \{x = (x', x_n) \in B_1 : x' \in A'_k\}.$$

Then A'_k is a subset of $B'(1)$ with C^2 boundary, and $A'_k \rightarrow B'(1)$ in the Hausdorff sense as $k \rightarrow \infty$.

We now claim that if k is large enough, then there exists a function $v_k \in C^{1,\alpha}(\bar{A}'_k)$ such that $v_k \leq \omega_k$ in Ω , $v_k = \omega_k$ on $\partial\Omega$, and

$$v_k \leq \omega_k \text{ in } A'_k, \quad v_k = \omega_k \text{ on } \partial A'_k, \quad Z_k \cap A_k = \{(x', x_n) \in A_k : x_n < v_k(x')\}. \quad (59)$$

Toward this end, we first note that standard compactness results imply that there exists some $Z \subset \mathbb{R}^n$ such that after passing to a subsequence if necessary, $\chi_{Z_k} \rightarrow \chi_Z$ in L^1_{loc} as $k \rightarrow \infty$. Then standard lowersemicontinuity results and the optimality of Z_k imply that, if we write $\varphi_0(\xi) := \varphi(0, \xi)$ and $H^- := \{(x', x_n) : x_n < 0\}$, then

$$\begin{aligned} P_{\varphi_0}(Z; B(2)) &\leq \liminf_{k \rightarrow \infty} P_{\varphi_k}(\Omega_k; B(2)) \leq \liminf_{k \rightarrow \infty} P_{\varphi_k}(Z_k; B(2)) \\ &= P_{\varphi_0}(H^-, B(2)). \end{aligned}$$

Also, since $Z_k = \Omega_k$ outside $B(1)$, it is clear that $Z \setminus B(1) = H^- \setminus B(1)$. However, convexity properties of φ_0 , see C_3 , imply that a half-plane is always strictly φ_0 -area minimizing with respect to compactly supported perturbations, so it follows that $Z = H^-$.

Once this is known, our claim about the existence of functions v_k , for k large enough, satisfying (59) follows from [4, 21]. More precisely, the proof of [21, Theorem 1] shows that the regularity theory for almost-minimizing currents, together with the fact that $\chi_{Z_k} \rightarrow \chi_{H^-}$ in L^1_{loc} , implies the claim. And the specific regularity results needed in our setting (where φ_k and the boundary data for Z_k depend on k , but are uniformly bounded in suitable norms as $k \rightarrow \infty$) are established in [4, Theorem 6.1].

For $w \in C^\infty(A'_k)$ such that $w = \omega_k$ on $\partial A'_k$ and $\text{graph}(w) \subset A_k \subset B(1)$, we define

$$I_k[w] := \int_{A'_k} \varphi_k(x', w, -Dw, 1) dx'.$$

Then $I_k[w] = P_{\varphi_k}(\{(x', x_n) \in A_k : x_n < w(x')\})$ for such w . It then follows from the optimality of Z_k that

$$I_k[v_k] \leq I_k[v_k - tw] \quad \text{for } w \in C^\infty(A'_k)_c \text{ such that } w(x) \geq 0 \text{ everywhere}$$

if t is positive and sufficiently small. It follows that if $w \geq 0$ with support in A'_k , then

$$0 \geq \lim_{t \searrow 0} \frac{1}{t} (I_k[v_k] - I_k[v_k - tw]) = \int_{A'_k} \varphi_{k,x_n}(x', v_k, -Dv_k, 1)w - \sum_{i=1}^{n-1} \varphi_{k,\xi_i}(x', v_k, -Dv_k, 1)w_{x_i} dx'.$$

Thus v_k satisfies

$$- \sum_{i=1}^{n-1} \partial_{x_i} \varphi_{k,\xi_i}(x', v_k, -Dv_k, 1) - \varphi_{k,x_n}(x', v_k, -Dv_k, 1) \geq 0 \quad (60)$$

weakly. On the other hand, (29) implies that

$$- \sum_{i=1}^{n-1} \partial_{x_i} \varphi_{k,\xi_i}(x', \omega_k, -D\omega_k, 1) - \varphi_{k,x_n}(x', \omega_k, -D\omega_k, 1) \leq 0, \quad (61)$$

with strict inequality on a dense subset of A'_k . (We recall the proof of this below, for the convenience of the reader.) If we let $w = v_k - \omega_k$, then arguing as in the proof of Lemma 4.1, we find that w is a weak subsolution a linear elliptic problem of the form (33), i.e. that

$$\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i} (-\langle \varphi_{k,\xi_i \xi_j} \rangle w_{x_j} + \langle \varphi_{k,\xi_i x_n} \rangle w) + \langle \varphi_{k,x_n \xi_i} \rangle w_{x_i} - \langle \varphi_{k,x_n x_n} \rangle w \leq 0,$$

weakly in A'_k . Moreover, (59) implies that $w \leq 0$ in A'_k , and $w = 0$ on $\partial A'_k$.

Our assumption (58) implies that $w = 0$ at some point $y_k \in A'_k$, and the weak Harnack inequality for subsolutions then implies that $w = 0$ everywhere in a small ball about y_k . This implies that $v_k = \omega_k$ in this small ball, which is impossible, in view of (60), together with the fact that (61) holds with strict inequality on a dense subset. This contradiction completes the proof of the lemma. \square

Finally we prove that (29) and (61) are equivalent. To show this, it suffices to prove that if w is a C^2 function $\mathbb{R}^{n-1} \rightarrow \mathbb{R}$ and if $d : \mathbb{R}^n \rightarrow \mathbb{R}$ is the signed distance to the graph of w (positive below the graph and negative above) then

$$\sum_{i=1}^n \partial_{x_i} \varphi_{\xi_i}(x, Dd) = \sum_{a=1}^{n-1} \partial_{x_a} \varphi_{\xi_a}(x', w, -Dw, 1) + \varphi_{x_n}(x', w, -Dw, 1). \quad (62)$$

at points $x = (x', w(x'))$ in the graph of w .

First, since $\varphi(x, \lambda \xi) = \lambda \varphi(x, \xi)$ for all x , it follows that $\varphi_{x_n}(x, \lambda \xi) = \lambda \varphi_{x_n}(x, \xi)$, and hence that

$$(-Dw, 1) \cdot \nabla_{\xi} \varphi_{x_n}(x', w, -Dw, 1) = \varphi_{x_n}(x', w, -Dw, 1).$$

Using this we see that

$$\begin{aligned} & \sum_{a=1}^{n-1} \partial_{x_a} \varphi_{\xi_a}(x', w, -Dw, 1) + \varphi_{x_n}(x', w, -Dw, 1) \\ &= \sum_{i,j=1}^n \varphi_{x_i \xi_i}(x', w, -Dw, 1) - \sum_{a,b=1}^{n-1} \varphi_{\xi_a \xi_b}(x', w, -Dw, 1) w_{x_a x_b}. \end{aligned}$$

Comparing with (62), we find that it now suffices to prove that

$$\sum_{i,j=1}^n \varphi_{\xi_i \xi_j}(x, Dd) d_{x_i x_j} = \sum_{a,b=1}^{n-1} \varphi_{\xi_a \xi_b}(x', w, -Dw, 1) w_{x_a x_b}.$$

To do this, it is helpful to define $f(x) = x_n - w(x')$. Then the zero level sets of d and of f coincide, so $Df = \frac{|Df|}{|Dd|} Dd = |Df| Dd$. Then by homogeneity, writing $x = (x', w(x'))$,

$$\varphi_{\xi_a \xi_b}(x', w, -Dw, 1) = \varphi_{\xi_a \xi_b}(x, Df) = |Df|^{-1} \varphi_{\xi_a \xi_b}(x, Dd)$$

and thus, using the form of f and the fact that $\partial_{\xi_i \xi_j}(x, \nu) \nu_j = 0$ (another consequence of homogeneity) we rewrite

$$\sum_{a,b=1}^{n-1} \varphi_{\xi_a \xi_b}(x', w, -Dw, 1) w_{x_a x_b} = \sum_{i,j=1}^n \varphi_{\xi_i \xi_j}(x, Dd) \frac{f_{x_i x_j}}{|Df|} = \sum_{i,j=1}^n \varphi_{\xi_i \xi_j}(x, Dd) \partial_{x_j} \left(\frac{f_{x_i}}{|Df|} \right).$$

However, $Dd = \frac{Df}{|Df|}$ at points in the graph of w (ie, the zero level-set of both d and f) which implies that all tangential derivatives of Dd and $Df/|Df|$ are equal. Then, again appealing to the fact that $\partial_{\xi_i \xi_j}(x, \nu) \nu_j = 0$, we conclude that

$$\sum_{i,j=1}^n \varphi_{\xi_i \xi_j}(x, Dd) \partial_{x_j} \left(\frac{f_{x_i}}{|Df|} \right) = \sum_{i,j=1}^n \varphi_{\xi_i \xi_j}(x, Dd) d_{x_i x_j},$$

completing the proof of (62). □

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